Lecture 5

Sampling

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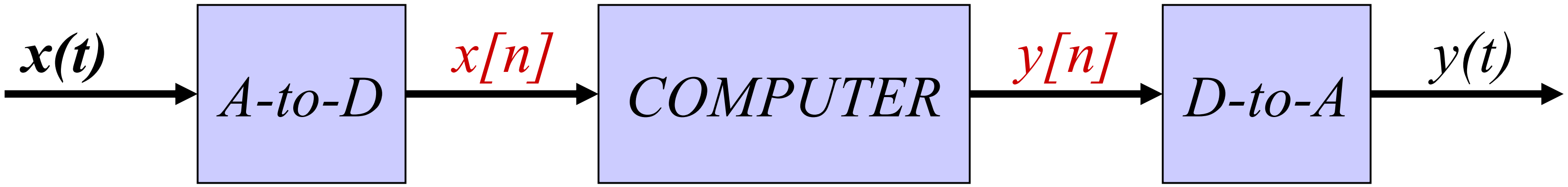
Continuous time vs Discrete time

- Continuous time system
  - Good for analogue & general understanding
  - Appropriate mostly to analogue electronic systems

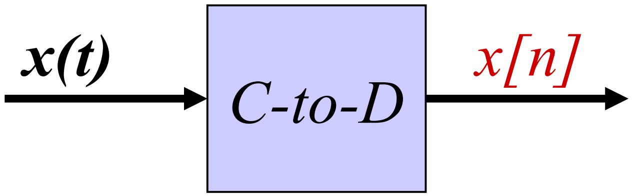
- Electronics are increasingly digital
  - E.g. mobile phones are all digital, TV broadcast is will be 100% digital in UK
  - We use digital ASIC chips, FPGAs and microprocessors to implement systems and to process signals
  - Signals are converted to numbers, processed, and converted back

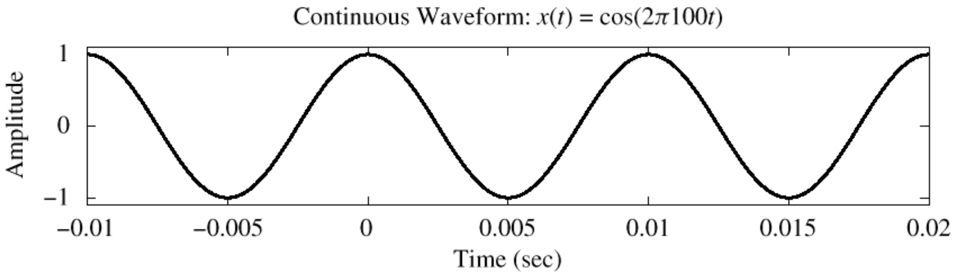


Sampling Process

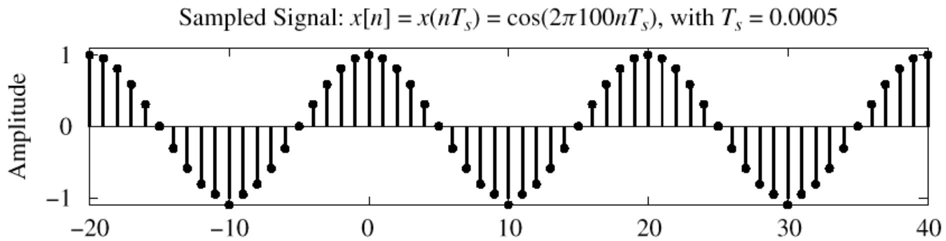
- Use A-to-D converter to turn x(t) into numbers x[n]
- Take a sample every sampling period T_s – uniform sampling

$x[n] = x(nT_s)$

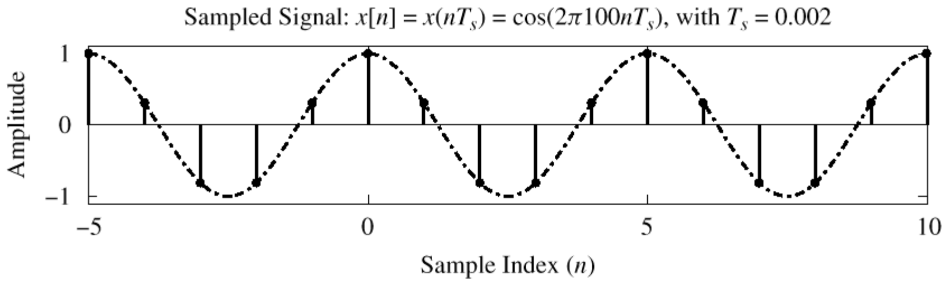




signal frequency $f = 100\text{Hz}$



sampling frequency $f_s = 2\text{kHz}$



sampling frequency $f_s = 500\text{Hz}$

Sampling Theorem

- Bridge between continuous-time and discrete-time
- Tell us HOW OFTEN WE MUST SAMPLE in order not to loose any information

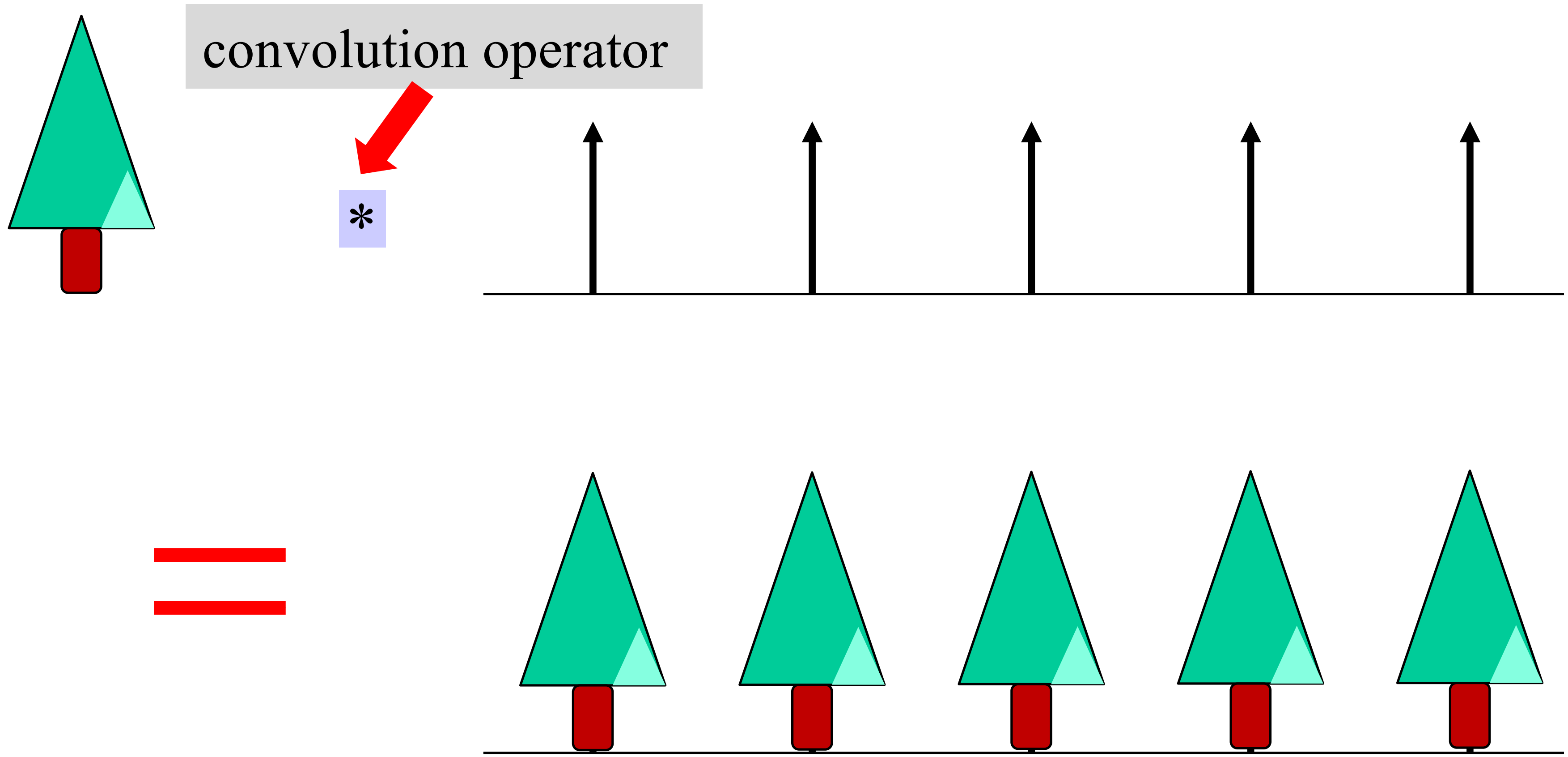
Sampling Theorem

A continuous-time signal $x(t)$ with frequencies no higher than f_{max} (Hz) can be reconstructed EXACTLY from its samples $x[n] = x(nT_s)$, if the samples are taken at a rate $f_s = 1/T_s$ that is greater than $2f_{max}$.

- For example, the sinewave on previous slide is 100 Hz. We need to sample this at higher than 200 Hz (i.e. 200 samples per second) in order NOT to loose any data, i.e. to be able to **reconstruct** the 100 Hz sinewave exactly.
- fmax refers to the maximum frequency component in the signal that has **significant** energy.
- Consequence of violating sampling theorem is **corruption of the signal** in digital form.

Intuitive idea of convolution

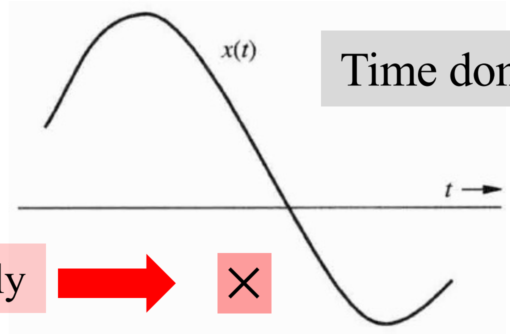
- Convolution – an important concept in signal processing
- Example: planting tree in a row, at regular interval



Sampling Theorem: Intuitive proof (1)

T_s

- Consider a band limited signal x(t) and has a spectrum X(ω):

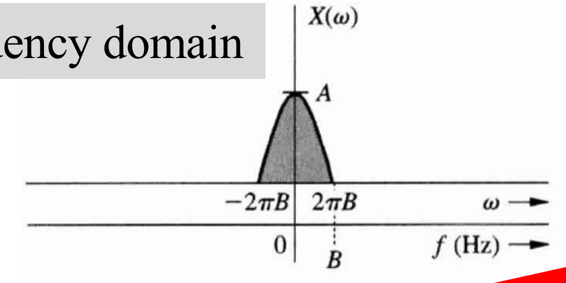


Time domain

FT

$\Longleftrightarrow$

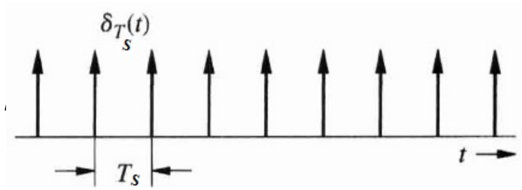
Frequency domain



multiply $\times$

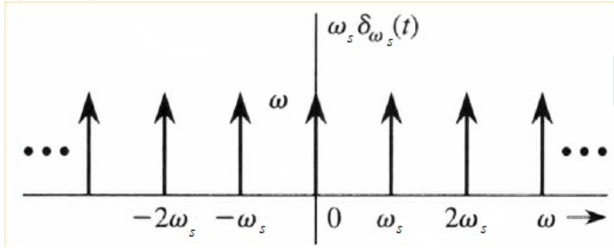
convolve $*$

- Ideal sampling = multiply x(t) with impulse train :



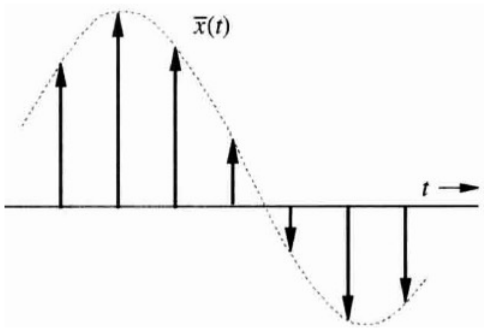
$\Longleftrightarrow$

$T_s \uparrow \;\Leftrightarrow\; \omega_s \downarrow$

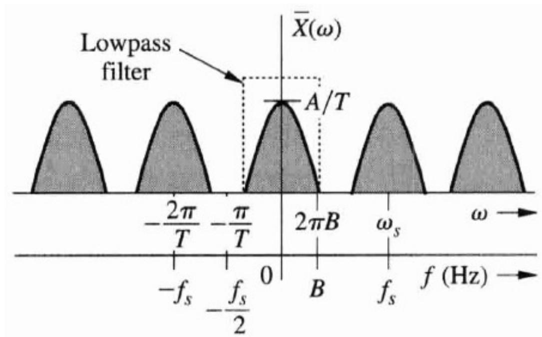


$\omega_s = 2\pi/T_s$

- Therefore the sampled signal has a spectrum (convolution):

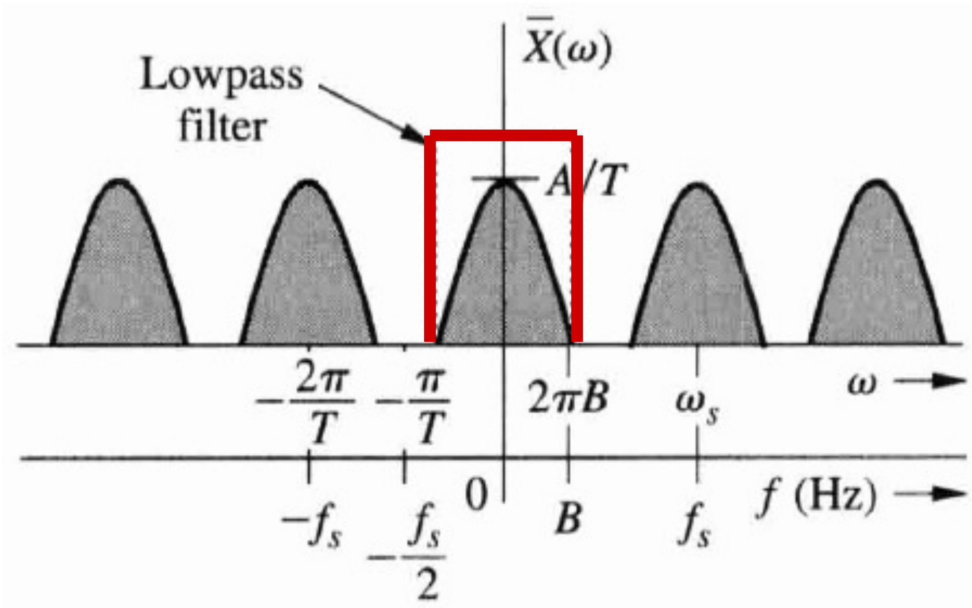


$\Longleftrightarrow$



Sampling Theorem: Intuitive proof (2)

- Therefore, to reconstruct the original signal x(t), we can use an ideal lowpass filter on the sampled spectrum:

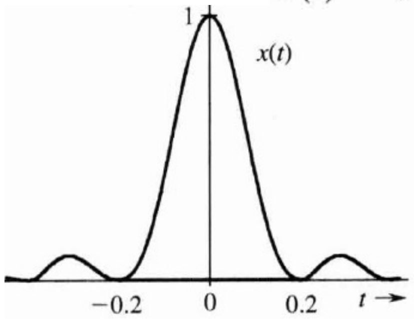


- This is only possible if the shaded parts do not overlap. This means that f_S must be more than TWICE that of B.

What happens if we sample too slowly? (1)

- What are the effects of sampling a signal at, above, and below the Nyquist rate? Consider a signal bandlimited to 5Hz:

$$x(t) = \text{sinc}^2(5\pi t) \quad \Leftrightarrow \quad X(\omega) = 0.2\,\Delta(\omega/20\pi)$$



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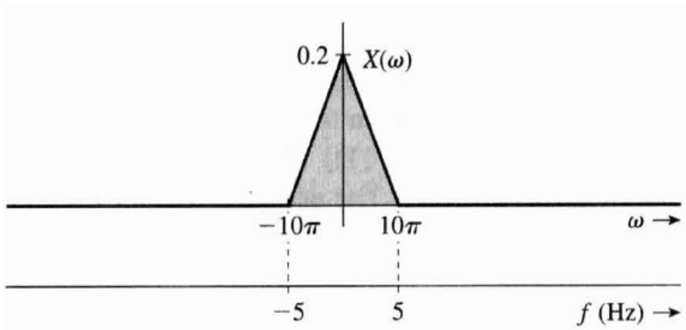
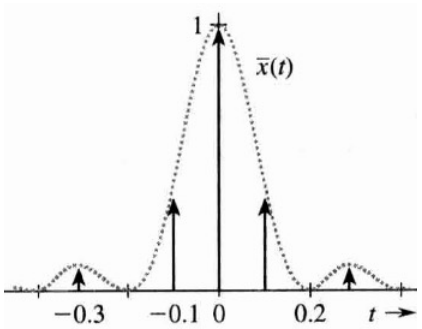


- Sampling at Nyquist rate of 10Hz give:

perfect reconstruction possible



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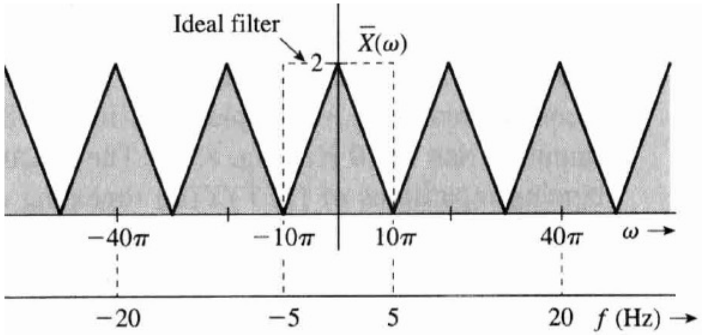
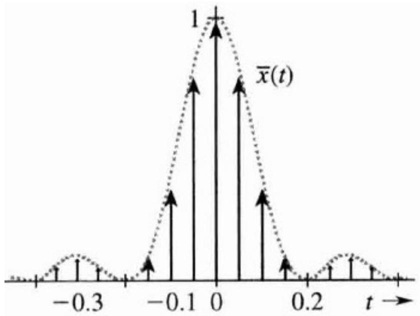


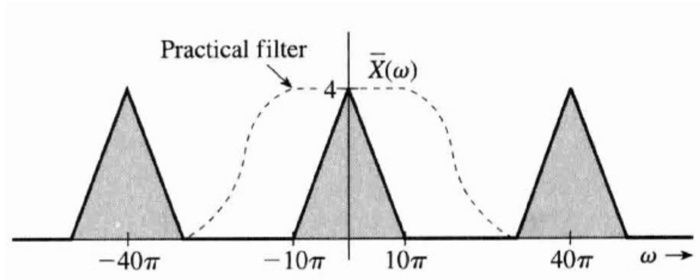
What happens if we sample too slowly? (2)

- Sampling at higher than Nyquist rate at 20Hz makes reconstruction much easier.

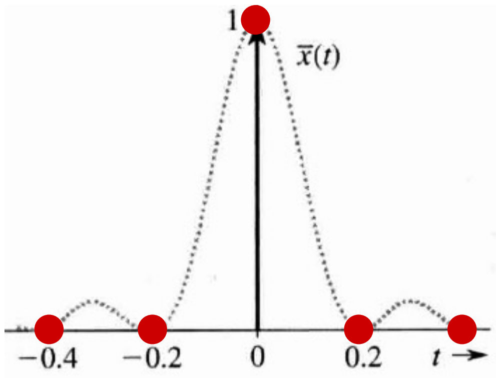
perfect reconstruction practical



$\Longleftrightarrow$

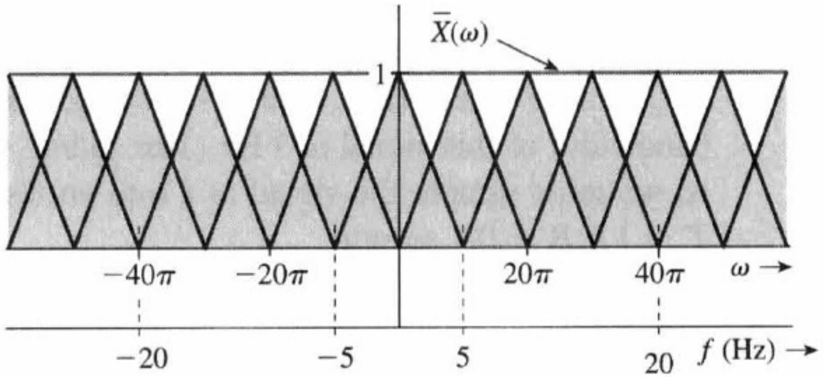


- Sampling below Nyquist rate at 5Hz corrupts the signal.



ALIASING

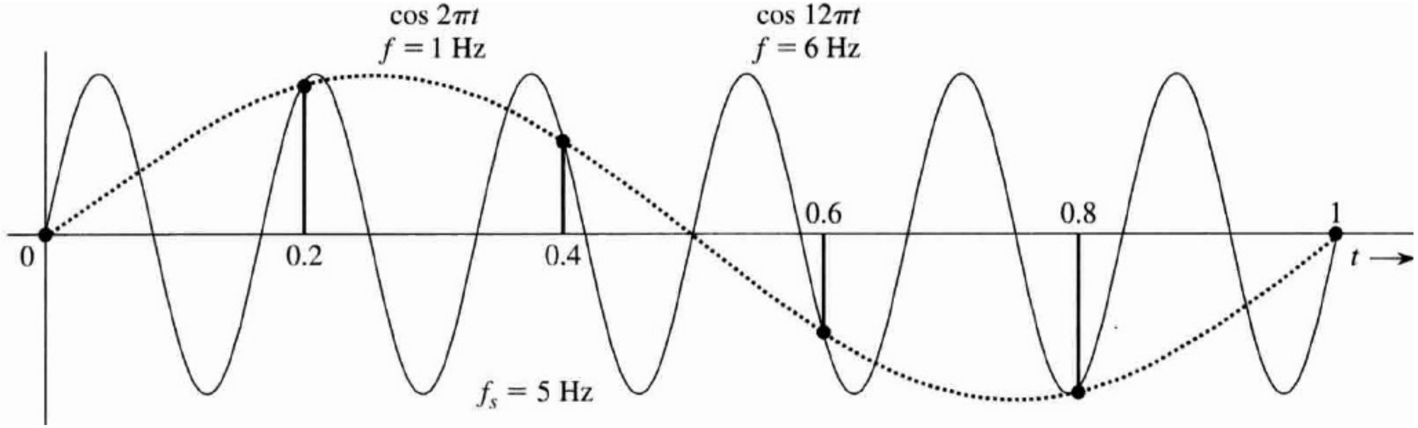
$\Longleftrightarrow$



Spectral folding effect of Aliasing

- Consider what happens when a 1Hz and a 6Hz sinewave is sampled at a rate of 5Hz.

1Hz & 6Hz sinewaves are indistinguishable after sampling



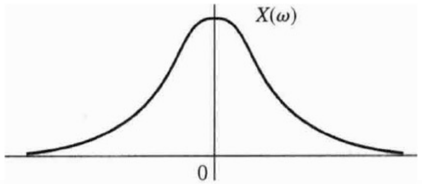
- In general, if a sinusoid of frequency f Hz is sampled at fs samples/sec, then sampled version would appear as samples of a continuous-time sinusoid of frequency $|f_a|$ in the band 0 to fs/2, where:

$$|f_a| = |f \pm m f_s| \qquad \text{where} \quad |f_a| \le \frac{f_s}{2}, \ \text{m is an integer}$$

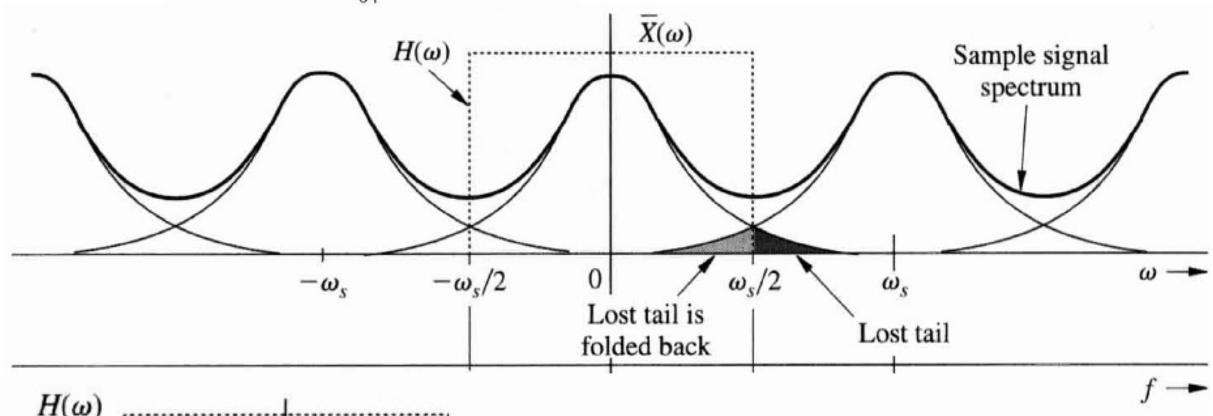
- In other words, the 6Hz sinusoid is FOLDED to 1Hz after being sampled at 5Hz.

Anti-aliasing filter (1)

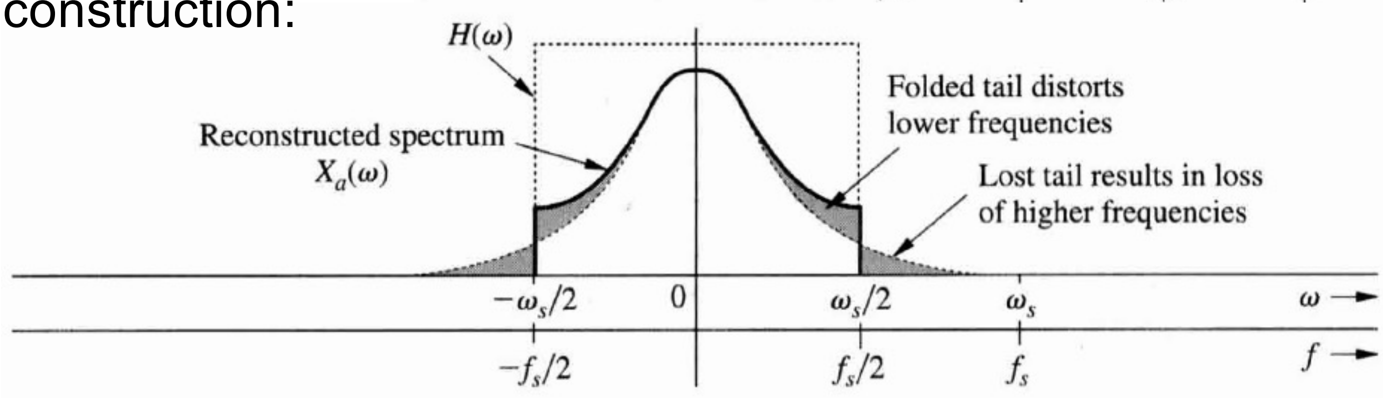
- To avoid corruption of signal after sampling, one must ensure that the signal being sampled at fs is bandlimited to a frequency B, where B < fs/2.
- Consider this signal spectrum:



- After sampling:

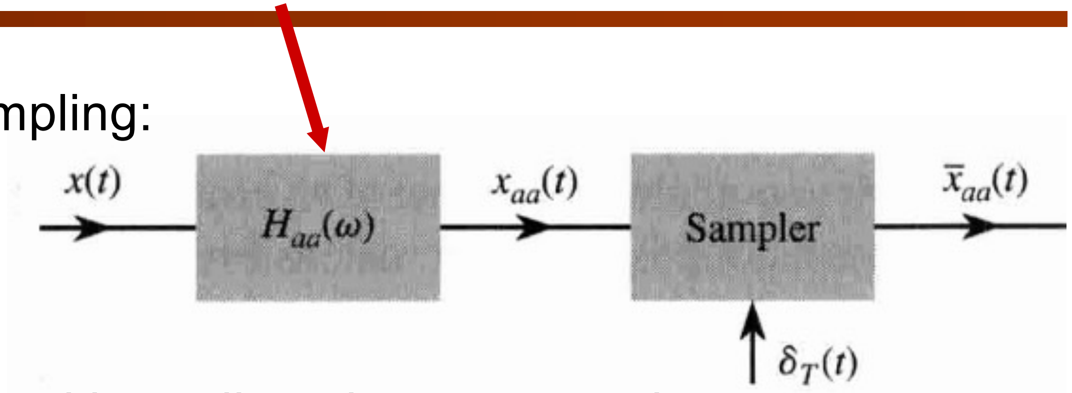


- After reconstruction:

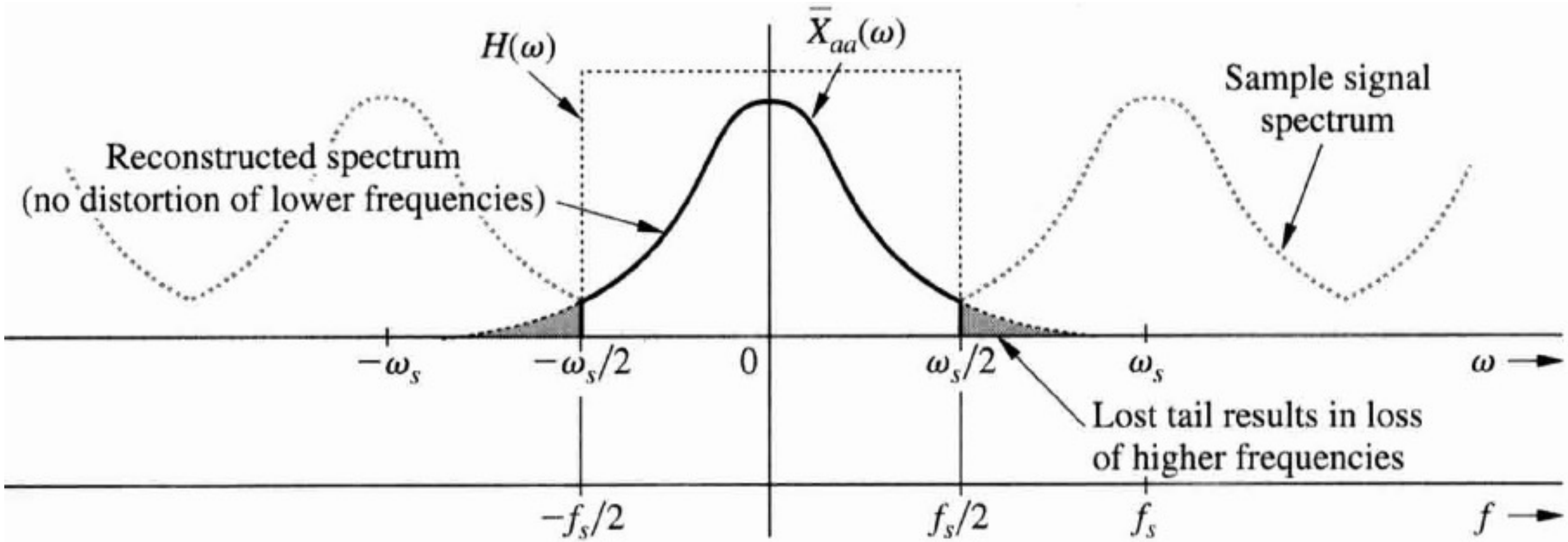


Anti-aliasing filter (2)

- Apply a lowpass filter before sampling:

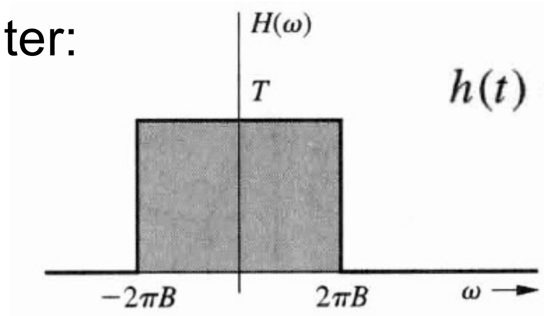


- Now reconstruction can be done without distortion or corruption to lower frequencies:

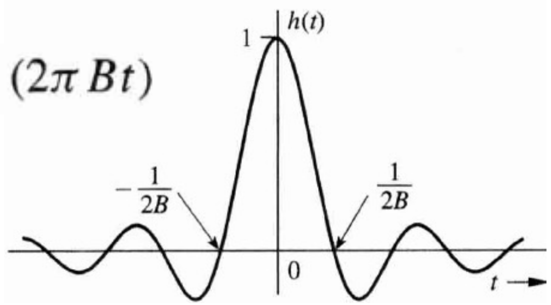


Ideal Signal Reconstruction

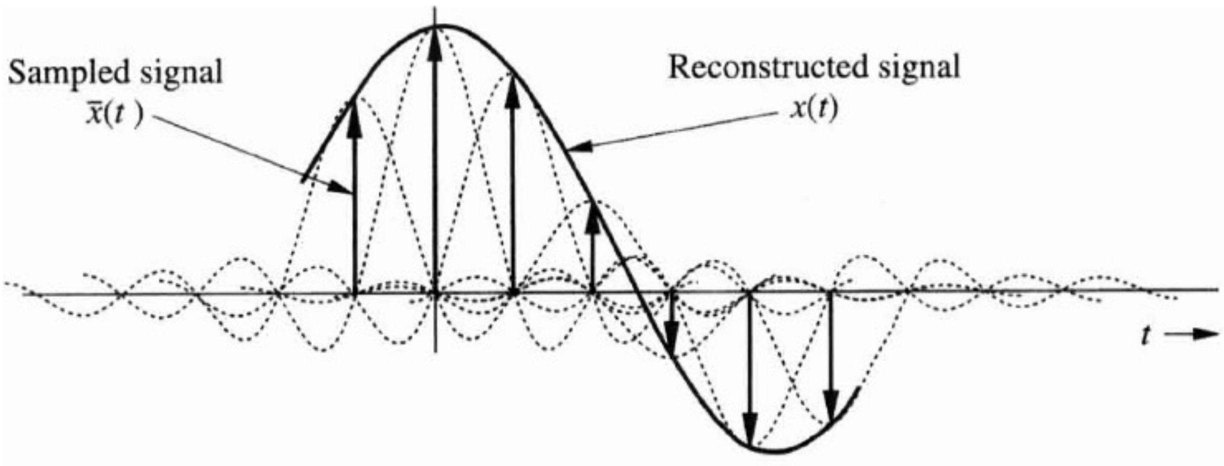
- Use ideal lowpass filter:



$$h(t) = \text{sinc}\,(2\pi Bt)$$



- That's why the sinc function is also known as the **interpolation** function:

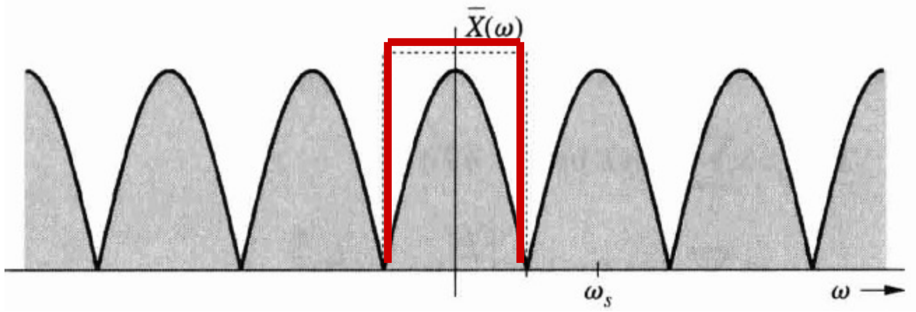


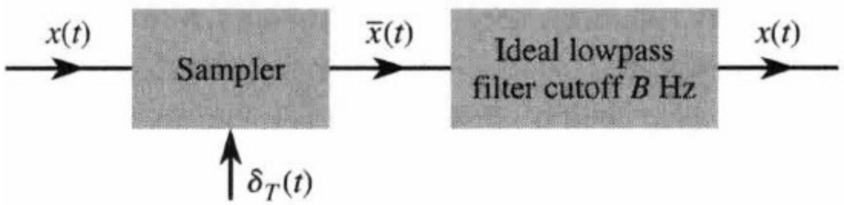
$$x(t) = \sum_n x(nT)h(t - nT)$$

$$= \sum_n x(nT)\,\text{sinc}\,(2\pi Bt - n\pi)$$

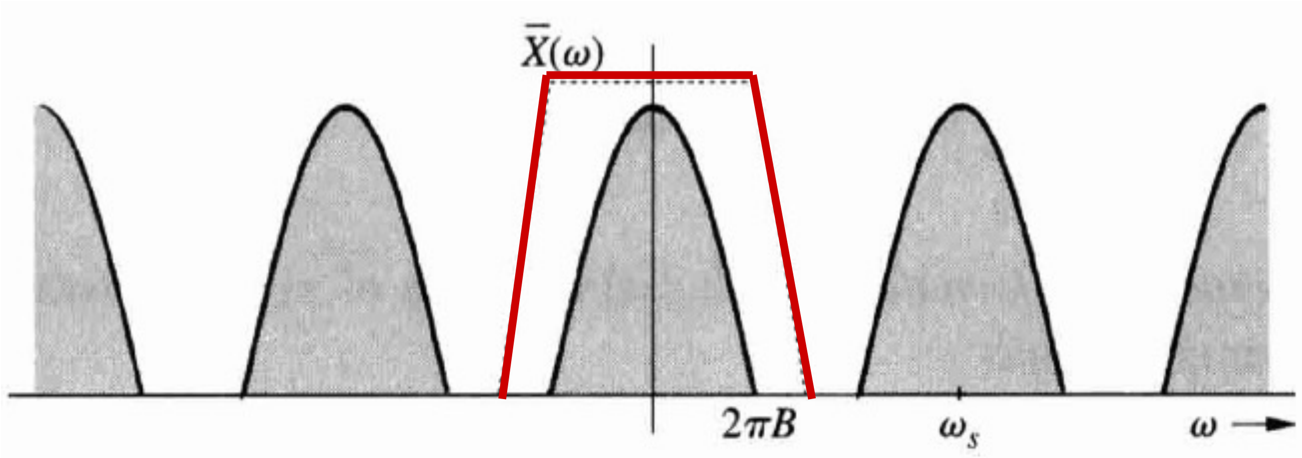
Practical Signal Reconstruction

- Ideal reconstruction system is therefore:



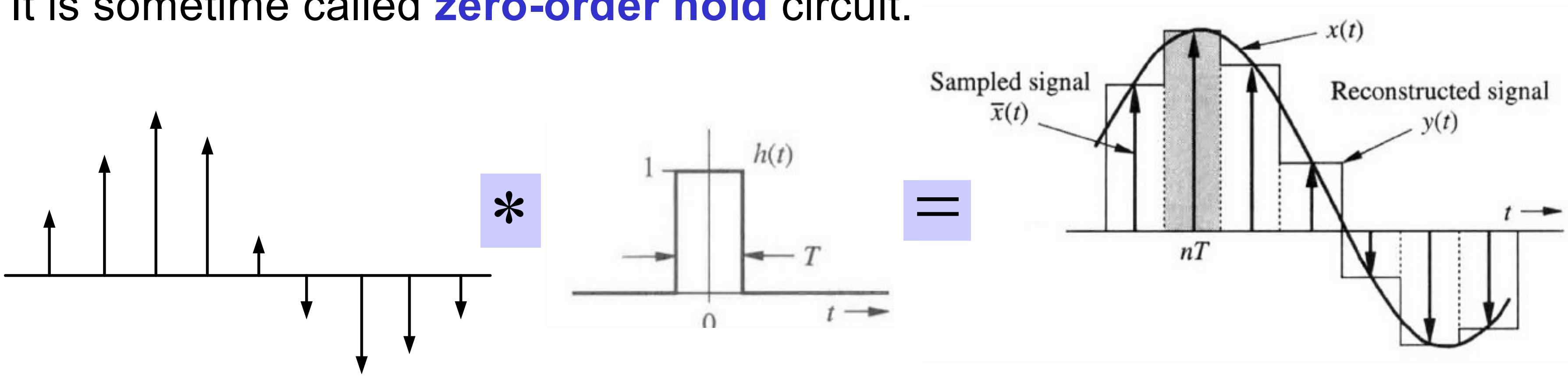


- In practice, we normally sample at higher frequency than twice f_{max}:

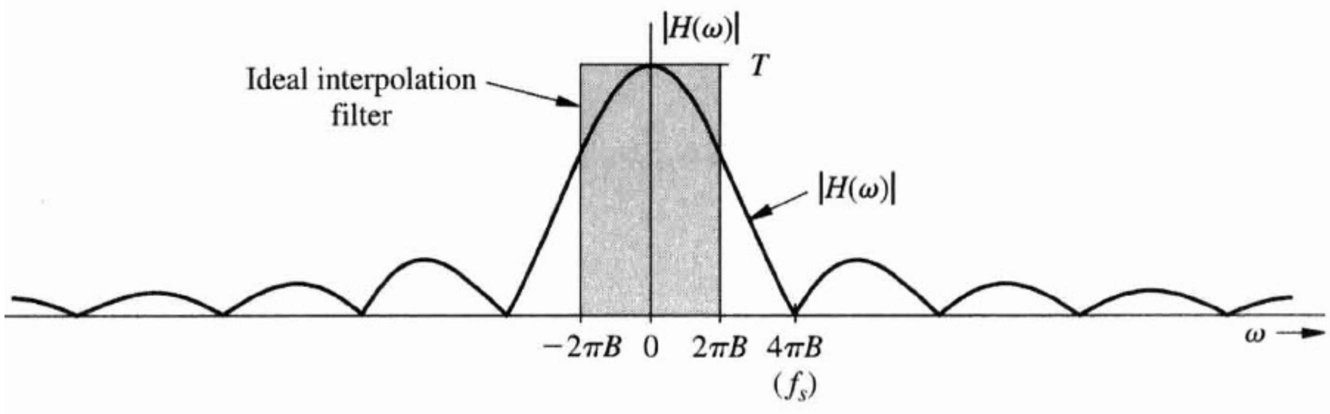


Signal Reconstruction using D/A converter

- D/A converter is a simple interpolator that performs the job of signal reconstruction.
- It is sometime called **zero-order hold** circuit.



- The effect of using the D/A converter is a non-ideal lowpass filter.



Three Big Ideas

1. **Sampling Theorem** tells us that we MUST sample a signal at a frequency that is higher than **TWICE the maximum signal frequency** to avoid corruption of the signal.

2. **Multiplication** in the **time domain** is the same as **convolution** in the **frequency domain**.

3. **Sampling changes the frequency spectrum** of the original signal – it introduces duplicate spectra of the original at 2fs, 3fs ……